Pergamon

Computers Math. Applic. Vol. 33, No. 7, pp. 85–99, 1997

0898-1221/97 $17.00 + 0.00

PII: S0898-1221(97)00044-8

Boundary Value Problems for General Discrete Systems on Infinite Intervals

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(Received and accepted October 1996)

Abstract—This paper continues the study of boundary value problems for a discrete system which in particular includes the prototype equation $x(k+1) = f(k, x(k))$, equations with finite as well as infinite delays, equations of neutral type, and the discrete integral equations of Volterra type. While the finite discrete interval case (regular as well as at resonance) has been discussed in [1,2], in the present paper the infinite interval case will be addressed.

Keywords—Generalized discrete system, Boundary value problem, Infinite interval, Existence theorems.

1. INTRODUCTION

Let $\mathbb{N} = \{0, 1, 2, \dots\}$ denote the set of nonnegative integers, and $x : \mathbb{N} \to \mathbb{R}^n$ with $x(k) = (x^1(k), \dots, x^n(k))$. Consider the discrete system

$$x(k+1) = \sum_{i=0}^{k} A_k(i)x(i) + b(k) + f_k(x(0), x(1), \dots, x(k)), \qquad k \in \mathbb{N}, \tag{1.1}$$

where each $A_k(i)$ is a constant $n \times n$ matrix; $b(k)$ is an n-vector, and $f_k : \mathbb{R}^{n(k+1)} \to \mathbb{R}^n$, with the dependence of f_k at k annotated in the subscript. The system (1.1) is very general, in fact, it includes in particular at least all the different types of equations listed in the abstract, and for this system recently in [3–7] a variety of problems have been discussed. Further, systems similar to (1.1) have been the subject matter of recent work in [8–11].

Let $B(\mathbb{N})$ be the space of all real n-vector bounded functions defined on $\mathbb{N}$, and let L be a bounded linear operator mapping $B(\mathbb{N})$ (or a subspace of $B(\mathbb{N})$) into $\mathbb{R}^n$. In this paper, first we shall study the discrete system (1.1) subject to the boundary conditions

$$L[x] = \ell \in \mathbb{R}^n. \tag{1.2}$$

In fact, in Section 2, we shall consider the problem (1.1),(1.2) with $f_k \equiv 0$, i.e., the linear problem and provide necessary and sufficient conditions for the existence of solutions. In Section 3, we shall apply various fixed point theorems to establish the existence of the solutions of the nonlinear

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX problem (1.1),(1.2). Then, in Section 4, we shall offer sufficient conditions for the existence of at least one value of the $\mathbb{R}^n$-valued parameter λ such that the discrete system

$$\begin{aligned} x(k+1) &= \sum_{i=0}^{k} A_k(i)x(i) + b(k) + g_k(x(0), x(1), \dots, x(k), \lambda), \qquad k \in \mathbb{N}, \\ x(0) &= \xi \end{aligned} \tag{1.3}$$

has a solution satisfying (1.2). For this, in what follows, throughout with respect to the difference systems (1.1) and (1.3), we shall assume that f_k and g_k are at least continuous in their domain of definitions.

The motivation of the present work comes from several studies for the boundary value problems similar to (1.1),(1.2), and (1.3),(1.2), and their continuous analogs over a finite interval in [12–36], and over infinite intervals in [37,38].

2. LINEAR PROBLEMS

Here, we shall provide necessary and sufficient conditions for the existence of solutions of the linear difference system

$$x(k+1) = \sum_{i=0}^{k} A_k(i)x(i) + b(k), \qquad k \in \mathbb{N}, \tag{2.1}$$

satisfying the boundary conditions (1.2). For this, the following two lemmas play a crucial role.

LEMMA 2.1. *(See [27].) For the linear system (2.1) together with the initial condition*

$$x(0) = \alpha, \tag{2.2}$$

the unique solution $x(k)$ can be written as

$$x(k) = \mathcal{A}(k)\alpha + \beta(k), \qquad k \in \mathbb{N}, \tag{2.3}$$

where

$$\beta(k) = \sum_{i=0}^{k-1} \mathcal{B}_k(i)b(i), \tag{2.4}$$

and the $n \times n$ matrices $\mathcal{A}(j)$ and $\mathcal{B}_k(i)$ are recursively defined as

$$\begin{aligned} \mathcal{A}(0) &= I, && \text{(identity)}, \\ \mathcal{A}(j+1) &= \sum_{i=0}^{j} A_j(i)\mathcal{A}(i), && 0 \le j \le k, \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} \mathcal{B}_k(i) &= 0, && i \ge k \ge 0, \\ \mathcal{B}_k(k-1) &= I, && k \ge 1, \\ \mathcal{B}_k(i) &= \sum_{\ell=i+1}^{k-1} A_{k-1}(\ell)\mathcal{B}_\ell(i), && 0 \le i < k-1. \end{aligned} \tag{2.6}$$

REMARK 2.1. For the remainder of this paper we shall assume that the rank of each of the matrices $\mathcal{A}(k)$, $k \in \mathbb{N}$, and $\mathcal{B}_k(i)$, $0 \le i \le k-1$, $k \in \mathbb{N}$ is n.

LEMMA 2.2. *(See [35].) Given a system of linear algebraic equations*

$$Ax = b, \tag{2.7}$$

where A is an $n \times n$ matrix and x and b are both n-dimensional vectors, suppose that the rank of A is $n - m$ $(1 \leq m \leq n)$.

The linear algebraic system (2.7) possesses a solution if and only if

$$\Theta b = 0, \tag{2.8}$$

where Θ is an $m \times n$ matrix whose row vectors are linearly independent vectors d_ζ, $1 \leq \zeta \leq m$, satisfying

$$d_\zeta A = 0. \tag{2.9}$$

In case (2.8) holds, any solution of (2.7) can be written as

$$x = \sum_{\zeta=1}^{m} \gamma_\zeta c_\zeta + Sb, \tag{2.10}$$

where γ_ζ, $1 \leq \zeta \leq m$, are arbitrary constants; c_ζ, $1 \leq \zeta \leq m$, are m linearly independent column vectors satisfying

$$Ac_\zeta = 0, \tag{2.11}$$

and S is an $n \times n$ matrix independent of b such that

$$ASp = p \tag{2.12}$$

for any column vector p satisfying

$$\Theta p = 0. \tag{2.13}$$

REMARK 2.2. The matrix S specified in Lemma 2.2 is not unique.

THEOREM 2.3. *A necessary and sufficient condition for the existence of a unique solution of the boundary value problem (2.1),(1.2) is that the matrix*

$$G = L\left[\mathcal{A}(k)\right] \tag{2.14}$$

is nonsingular. Further, this solution $x(k)$ can be represented as

$$x(k) = H_1[b(k)] + H_2[\ell], \tag{2.15}$$

where H_1 is the linear operator mapping $B(\mathbb{N})$ into itself such that

$$H_1[b(k)] = \sum_{i=0}^{k-1} \mathcal{B}_k(i)b(i) - \mathcal{A}(k)G^{-1}L\left[\sum_{i=0}^{k-1} \mathcal{B}_k(i)b(i)\right],$$

and H_2 is the linear operator mapping $\mathbb{R}^n$ into $B(\mathbb{N})$ such that

$$H_2[\ell] = \mathcal{A}(k)G^{-1}\ell.$$

PROOF. The solution (2.3) satisfies (1.2) if and only if

$$L[\mathcal{A}(k)]\alpha + L[\beta(k)] = \ell. \tag{2.16}$$

Since $\det G \neq 0$, from (2.16) we get

$$\alpha = G^{-1}\ell - G^{-1}L[\beta(k)]. \tag{2.17}$$

On substituting (2.17) in (2.3), the result (2.15) follows. ∎

COROLLARY 2.4. *A necessary and sufficient condition for the existence of a unique solution of the boundary value problem: (2.1),*

$$\sum_{s=1}^{K} L_s x(k_s) = \ell, \qquad 0 = k_1 < k_2 < \cdots < k_K = \infty \quad (2 \le K < \infty), \tag{2.18}$$

where L_s, $1 \le s \le K$ are $n \times n$ matrices, is that the matrix

$$G_1 = \sum_{s=1}^{K} L_s \mathcal{A}(k_s) \tag{2.19}$$

is nonsingular. Further, this solution $x(k)$ can be represented as

$$x(k) = \mathcal{A}(k) G_1^{-1} \ell + \sum_{i=0}^{\infty} g(k,i) b(i), \tag{2.20}$$

where $g(k,i)$ is the Green's matrix such that for $k_{s-1} \le i \le k_s - 1$, $2 \le s \le K$

$$g(k,i) = \begin{cases} \mathcal{B}_k(i) - \mathcal{A}(k) G_1^{-1} \displaystyle\sum_{j=s}^{K} L_j \mathcal{B}_{k_j}(i), & k_{s-1} \le i \le k-1, \\ -\mathcal{A}(k) G_1^{-1} \displaystyle\sum_{j=s}^{K} L_j \mathcal{B}_{k_j}(i), & k \le i \le k_s - 1. \end{cases} \tag{2.21}$$

PROOF. For the boundary conditions (2.18) the equation (2.17) becomes

$$\alpha = G_1^{-1} \ell - G_1^{-1} \sum_{s=1}^{K} L_s \sum_{i=0}^{k_s - 1} \mathcal{B}_{k_s}(i) b(i),$$

which on arranging the terms is the same as

$$\alpha = G_1^{-1} \ell - G_1^{-1} \sum_{s=2}^{K} \sum_{i=k_{s-1}}^{k_s - 1} \sum_{j=s}^{K} L_j \mathcal{B}_{k_j}(i) b(i),$$

and hence, the solution of (2.1),(2.18) is

$$x(k) = \mathcal{A}(k) G_1^{-1} \ell - \mathcal{A}(k) G_1^{-1} \sum_{s=2}^{K} \sum_{i=k_{s-1}}^{k_s - 1} \sum_{j=s}^{K} L_j \mathcal{B}_{k_j}(i) b(i) + \sum_{i=0}^{k-1} \mathcal{B}_k(i) b(i),$$

which from the definition of $g(k,i)$ is the same as (2.20). ∎

THEOREM 2.5. *Let the rank of the matrix G defined in (2.14) be $n - m$ $(1 \le m \le n)$. Then, the boundary value problem (2.1),(1.2) has a solution if and only if*

$$\Theta \ell - \Theta L \left[\sum_{i=0}^{k-1} \mathcal{B}_k(i) b(i) \right] = 0, \tag{2.22}$$

where Θ is an $m \times n$ matrix whose row vectors are linearly independent vectors d_ζ, $1 \le \zeta \le m$ satisfying $d_\zeta G = 0$.

In case (2.22) holds, any solution of (2.1),(1.2) can be expressed as

$$x(k) = \sum_{\zeta=1}^{m} \gamma_\zeta y_\zeta(k) + \overline{H}_1[b(k)] + \overline{H}_2[\ell], \tag{2.23}$$

where γ_ζ, $1 \le \zeta \le m$ are arbitrary constants; $y_\zeta(k)$, $1 \le \zeta \le m$ are m linearly independent solutions of the homogeneous boundary value problem

$$x(k+1) = \sum_{i=0}^{k} A_k(i)x(i), \qquad L[x] = 0; \tag{2.24}$$

$\overline{H}_1$ is the linear operator mapping $B(\mathbb{N})$ into itself such that

$$\overline{H}_1[b(k)] = \sum_{i=0}^{k-1} \mathcal{B}_k(i)b(i) - \mathcal{A}(k)SL\left[\sum_{i=0}^{k-1} \mathcal{B}_k(i)b(i)\right],$$

and $\overline{H}_2$ is the linear operator mapping $\mathbb{R}^n$ into $B(\mathbb{N})$ such that

$$\overline{H}_2[\ell] = \mathcal{A}(k)S\ell.$$

The matrix S is an $n \times n$ matrix independent of $\ell - L\left[\sum_{i=0}^{k-1} \mathcal{B}_k(i)b(i)\right]$ such that $GSp = p$ for any column vector p satisfying $\Theta p = 0$.

PROOF. From Lemma 2.2 the system (2.16) has a solution if and only if (2.22) holds. Further, in such a case the vector α can be given by

$$\alpha = \sum_{\zeta=1}^{m} \gamma_\zeta c_\zeta + S\ell - SL\left[\sum_{i=0}^{k-1} \mathcal{B}_k(i)b(i)\right], \tag{2.25}$$

where c_ζ, $1 \le \zeta \le m$ are m linearly independent column vectors satisfying $Gc_\zeta = 0$. Let $y_\zeta(k) = \mathcal{A}(k)c_\zeta$, $1 \le \zeta \le m$ then in view of (2.3) and the Remark 2.1, $y_\zeta(k)$ are linearly independent solutions of the homogeneous difference system (2.24). Moreover, we have

$$L[y_\zeta(k)] = L[\mathcal{A}(k)c_\zeta] = L[\mathcal{A}(k)]c_\zeta = Gc_\zeta = 0, \qquad 1 \le \zeta \le m.$$

Now substituting (2.25) in (2.3), we find (2.23). ∎

COROLLARY 2.6. *Let the rank of the matrix G_1 defined in (2.19) be $n - m$ $(1 \le m \le n)$. Then, the boundary value problem (2.1),(2.18) has a solution if and only if*

$$\Theta\ell - \Theta\sum_{s=2}^{K} L_s \sum_{i=0}^{k_s-1} \mathcal{B}_{k_s}(i)b(i) = 0, \tag{2.26}$$

where Θ is an $m \times n$ matrix whose row vectors are linearly independent vectors d_ζ, $1 \le \zeta \le m$ satisfying $d_\zeta G_1 = 0$.

In case (2.26) holds, any solution of (2.1),(2.18) can be expressed as

$$x(k) = \sum_{\zeta=1}^{m} \gamma_\zeta y_\zeta(k) + \mathcal{A}(k)S\ell + \sum_{i=0}^{\infty} g_1(k,i)b(i), \tag{2.27}$$

where γ_ζ, $1 \le \zeta \le m$ are arbitrary constants; $y_\zeta(k)$, $1 \le \zeta \le m$ are m linearly independent solutions of the homogeneous boundary value problem

$$x(k+1) = \sum_{i=0}^{k} A_k(i)x(i), \qquad \sum_{s=1}^{K} L_s x(k_s) = 0; \tag{2.28}$$

S is an $n \times n$ matrix independent of $\ell - \sum_{s=2}^{K} L_s \sum_{i=0}^{k_s-1} \mathcal{B}_{k_s}(i)b(i)$ such that $GSp = p$ for any column vector p satisfying $\Theta p = 0$, and $g_1(k,i)$ is the Green's matrix such that for $k_{s-1} \le i \le k_s - 1$, $2 \le s \le K$

$$g_1(k,i) = \begin{cases} \mathcal{B}_k(i) - \mathcal{A}(k)S\displaystyle\sum_{j=s}^{K} L_j\mathcal{B}_{k_j}(i), & k_{s-1} \le i \le k-1, \\ -\mathcal{A}(k)S\displaystyle\sum_{j=s}^{K} L_j\mathcal{B}_{k_j}(i), & k \le i \le k_s - 1. \end{cases} \tag{2.29}$$

PROOF. The proof is similar to that of Corollary 2.4 and Theorem 2.5. ∎

REMARK 2.3. In view of Remark 2.2 the matrix S in Theorem 2.5 and Corollary 2.6 is not unique.

3. NONLINEAR PROBLEMS

In the space $B(\mathbb{N})$ we shall consider the norm $\|x\|_B = \sup_{k\in\mathbb{N}} \|x(k)\|$, where $\|\alpha\| = \sum_{i=1}^{n} |\alpha_i|$, $\alpha \in \mathbb{R}^n$. Let $B^\infty(\mathbb{N})$ consists of all functions $x \in B(\mathbb{N})$ for which $\lim_{k\to\infty} x(k)$ exists and is finite. It is clear that $B^\infty(\mathbb{N})$ is a closed subset of $B(\mathbb{N})$. For a fixed positive real number ν we define the set $S_\nu = \{\alpha \in \mathbb{R}^n : \|\alpha\| \le \nu\}$.

CONDITION C$_1$. $\lim_{k\to\infty} \mathcal{A}(k) = \mathcal{A}(\infty)$ and $\lim_{k\to\infty} \mathcal{B}_k(i) = \mathcal{B}_\infty(i)$ exist and are finite.

REMARK 3.1. In the rest of this paper we shall assume that there exist matrices $X(k)$ and $Y(i)$ defined for $0 \le i \le k \in \mathbb{N}$ such that $\mathcal{B}_k(i) = X(k)Y(i)$. This condition is satisfied by many difference systems, e.g., when $A_k(i) = 0$, $0 \le i \le k-1$ and $A_k(k) = A(k)$, say, then $\mathcal{A}(k) = \prod_{j=0}^{k-1} A(k-1-j)$, and $\mathcal{B}_k(i) = \mathcal{A}(k)\mathcal{A}^{-1}(i+1)$. With this assumption we note that the condition $\lim_{k\to\infty} \mathcal{B}_k(i) = \mathcal{B}_\infty(i)$ implies that $\lim_{k\to\infty} X(k) = X(\infty)$ exists and is finite. We shall also assume that the matrix G defined in (2.14) is nonsingular.

REMARK 3.2. In our main results the following proposition will be used. A set $Z \subset B^\infty(\mathbb{N})$ is relatively compact if it is bounded, and uniformly convergent in the following sense: for each $\epsilon > 0$ there exists a $k_\epsilon \in \mathbb{N}$ such that $\|x(k) - x(\infty)\| < \epsilon$ for every $k_\epsilon < k \in \mathbb{N}$ and $x \in Z$. It should be emphasized that k_ϵ is independent of the function $x(k)$. Although the proof of this proposition can be modelled after Avramescu [39], for completeness the proof is provided here. We need to show that every sequence in Z has a Cauchy subsequence. From any sequence in Z, of course, we can extract a subsequence, say $\{g_m\}$, such that $\{g_m(\infty)\}$ is convergent. Fix $\epsilon > 0$. There exists a positive integer k_ϵ with

$$\|g_m(k) - g_m(\infty)\| < \frac{\epsilon}{3}, \qquad \text{if } k > k_\epsilon,\ m \in \{1,2,\dots\} \text{ and } k \in \mathbb{N}.$$

Also there exists a positive integer h_ϵ with

$$\|g_m(\infty) - g_p(\infty)\| < \frac{\epsilon}{3}, \qquad \text{for } m,\ p \ge h_\epsilon \text{ and } m,\ p \in \mathbb{N}.$$

Combine these two inequalities, to obtain

$$\|g_m(k) - g_p(k)\| < \epsilon, \qquad \text{if } k > k_\epsilon \text{ and } m,\ p \ge h_\epsilon.$$

On the other hand, the Arzela-Ascoli theorem [18] implies that there exists a subsequence $\{g_{m(r)}\}$ of $\{g_m\}$ and a positive integer q_ϵ with

$$\|g_{m(r)}(i) - g_{m(h)}(i)\| < \epsilon, \qquad \text{for } i \in \{1,\dots,k_\epsilon\} \text{ with } m(r),\ m(h) \ge q_\epsilon,$$

and $m(r), m(h) \in \mathbb{N}$. The last two inequalities imply that the subsequence $\{g_{m(r)}\}$ in Z is Cauchy.

THEOREM 3.1. *With respect to the boundary value problem (1.1),(1.2) assume that the operator L is defined on $B^{\infty}(\mathbb{N})$ and Condition C_1 holds. Further suppose that there exists a positive real number ν with*

(i) $\max\{\sup_{k\in\mathbb{N}} \|\mathcal{A}(k)\|, \sup_{k\in\mathbb{N}} \|X(k)\|\} = Q$;
(ii) *if* $\sup_{x\in S^{\nu}} \|Y(i)\,[b(i) + f_i(x(0), x(1), \dots, x(i))]\| = q(i)$, *where* $S^{\nu} = \{x \in B^{\infty}(\mathbb{N}) : \|x\|_B \leq \nu\}$, *then* $\sum_{i=0}^{\infty} q(i) = P < \infty$;
(iii) *for every* $x \in S^{\nu}$,

$$\left\| G^{-1}\left(\ell - L\left[\sum_{i=0}^{k-1} \mathcal{B}_k(i)\,[b(i) + f_i(x(0), x(1), \dots, x(i))]\right]\right)\right\| \leq M;$$

(iv) $Q(M + P) \leq \nu$.

Then, there exists at least one solution of (1.1),(1.2) in S^{ν}.

PROOF. Consider the operator $T : S^{\nu} \to B^{\infty}(\mathbb{N})$ defined as follows:

$$\begin{aligned}(Tx)(k) = {} & X(k)\sum_{i=0}^{k-1} Y(i)\,[b(i) + f_i(x(0), x(1), \dots, x(i))] \\ & - \mathcal{A}(k)G^{-1}L\left[\sum_{i=0}^{k-1} \mathcal{B}_k(i)\,[b(i) + f_i(x(0), x(1), \dots, x(i))]\right] + \mathcal{A}(k)G^{-1}\ell.\end{aligned} \tag{3.1}$$

In view of Theorem 2.3 it is clear that a fixed point of T is a solution of the boundary value problem (1.1),(1.2).

For $x \in S^{\nu}$ conditions (i)–(iv) imply that

$$\|(Tx)\|_B \leq QM + Q\sum_{i=0}^{\infty} q(i) = Q(M + P) \leq \nu,$$

i.e., $TS^{\nu} \subseteq S^{\nu}$. Now fix $x \in S^{\nu}$ and let $y_x(k) = (Tx)(k)$ with $\lim_{k\to\infty} y_x(k) = y_x(\infty)$. Then, we have

$$\begin{aligned}\|y_x(k) - y_x(\infty)\| = {} & \Bigg\| X(k)\sum_{i=0}^{k-1} Y(i)\,[b(i) + f_i(x(0), x(1), \dots, x(i))] \\ & - \mathcal{A}(k)G^{-1}L\left[\sum_{i=0}^{k-1} \mathcal{B}_k(i)\,[b(i) + f_i(x(0), x(1), \dots, x(i))]\right] + \mathcal{A}(k)G^{-1}\ell \\ & - X(\infty)\sum_{i=0}^{\infty} Y(i)\,[b(i) + f_i(x(0), x(1), \dots, x(i))] \\ & + \mathcal{A}(\infty)G^{-1}L\left[\sum_{i=0}^{k-1} \mathcal{B}_k(i)\,[b(i) + f_i(x(0), x(1), \dots, x(i))]\right] - \mathcal{A}(\infty)G^{-1}\ell\Bigg\| \\ \leq {} & M\|\mathcal{A}(k) - \mathcal{A}(\infty)\| + P\|X(k) - X(\infty)\| + Q\sum_{i=k}^{\infty} q(i).\end{aligned}$$

Thus, it follows that given $\epsilon > 0$ there exists $k_0(\epsilon) \in \mathbb{N}$ such that $\|y_x(k) - y_x(\infty)\| < \epsilon$, for every $k_0(\epsilon) \leq k \in \mathbb{N}$ and every $x \in S^{\nu}$. Consequently, $\{y_x\}$, $x \in S^{\nu}$ is relatively compact in $B^{\infty}(\mathbb{N})$. To show the continuity of T on S^ν, we let $\{x_m\} \subset S^\nu$, such that $\lim_{m\to\infty} \|x_m - x\|_B \to 0$. Further, let $y_m(k) = (Tx_m)(k)$, $y(k) = (Tx)(k)$. Then, we obtain

$$\begin{aligned}\|y_m - y\|_B = &\left\| X(k) \sum_{i=0}^{k-1} Y(i) \left[f_i(x_m(0), \dots, x_m(i)) - f_i(x(0), \dots, x(i)) \right] \right. \\ &\left. -A(k)G^{-1}L \left[X(k) \sum_{i=0}^{k-1} Y(i) \left[f_i(x_m(0), \dots, x_m(i)) - f_i(x(0), \dots, x(i)) \right] \right] \right\| \\ &\le Q \left(1 + \|G^{-1}\| \, \|L\| \, Q\right) \sum_{i=0}^{\infty} \|Y(i) \left[f_i(x_m(0), \dots, x_m(i)) - f_i(x(0), \dots, x(i)) \right]\| .\end{aligned} \tag{3.2}$$

From the continuity of f_i, it is clear that the summand in the last term of (3.2) tends to zero as $m \to \infty$, further in view of (ii) it is uniformly bounded by the summable function $2q(i)$. Thus, it follows from Lebesgue's dominated convergence theorem that $\|y_m - y\|_B \to 0$ as $m \to \infty$, i.e., T is continuous on S^ν. The existence of a fixed point of T now follows as an application of Schauder's fixed point theorem. As we have noted earlier this fixed point is a solution of (1.1),(1.2). ∎

COROLLARY 3.2. *With respect to the boundary value problem (1.1),(1.2) assume that the operator L is defined on $B^\infty(\mathbb{N})$ and Conditions C_1 and (i) hold. Further, let*

(v) $$\liminf_{m\to\infty} \frac{1}{m} \sum_{i=0}^{\infty} \sup_{\substack{x \in B^\infty(\mathbb{N}) \\ \|x\|_B \le m}} \|Y(i) \left[b(i) + f_i(x(0), x(1), \dots, x(i)) \right]\| = 0.$$

Then, for every $\ell \in \mathbb{R}^n$ there exists at least one solution of (1.1),(1.2).

PROOF. Fix $\ell \in \mathbb{R}^n$ and choose a sequence $\{m_p\}$ of positive integers such that

$$\lim_{p\to\infty} m_p = \infty, \quad \text{and} \quad \lim_{p\to\infty} \lambda_p = 0,$$

where

$$\lambda_p = \frac{1}{m_p} \sum_{i=0}^{\infty} \sup_{\substack{x \in B^\infty(\mathbb{N}) \\ \|x\|_B \le m_p}} \|Y(i) \left[b(i) + f_i(x(0), x(1), \dots, x(i)) \right]\| .$$

Then, there exists a p_0 such that for every $p \ge p_0$,

$$\frac{1}{m_p} Q \|G^{-1}\| \, \|\ell\| + Q \left(\|G^{-1}\| \, \|L\| Q + 1 \right) \lambda_p \le 1.$$

Thus, for every $x \in S^\nu = \{x \in B^\infty(\mathbb{N}) : \|x\|_B \le m_{p_0} = \nu\}$ from (3.1) it follows that

$$\|(Tx)(k)\| \le m_{p_0} = \nu,$$

i.e., $TS^\nu \subseteq S^\nu$. The rest of the proof is similar to that of Theorem 3.1. ∎

From (3.1) it is clear that for any solution $x(k)$ of the boundary value problem (1.1),(1.2) the initial condition $x(0) = x_0$ has the following representation

$$x_0 = G^{-1} \left(\ell - L \left[X(k) \sum_{i=0}^{k-1} Y(i) \left[b(i) + f_i(x(0), x(1), \dots, x(i)) \right] \right] \right). \tag{3.3}$$

In our next result by $x(k, 0, u)$ we shall denote the solution $x(k)$ of (1.1) satisfying $x(0) = u$.

THEOREM 3.3. *With respect to the boundary value problem (1.1),(1.2) assume that the operator L is defined on $B(\mathbb{N})$ and Condition (i) holds. Further suppose that there exists a positive real number ν with*

(vi) *for each $u \in S_\nu$ there exists a (unique) $x(k,0,u)$ on $\mathbb{N}$;*

(vii) *there exists a constant $D > 0$ such that $\|x(k,0,u)\| \le D$, $u \in S_\nu$;*

(viii) *if $\sup_{\|x(k,0,u)\| \le D} \|Y(i)\,[b(i)+f_i(x(0,0,u),\dots,x(i,0,u))]\| = q(i)$, then $\sum_{i=0}^{\infty} q(i) = P < \infty$;*

(ix) $\|G^{-1}\|\,(\|\ell\| + \|L\|QP) \le \nu$.

Then, there exists at least one solution of (1.1),(1.2).

PROOF. Consider the operator $T : S_\nu \to \mathbb{R}^n$ defined as follows

$$Tu = G^{-1}\left(\ell - L\left[X(k)\sum_{i=0}^{k-1} Y(i)\,[b(i) + f_i(x(0,0,u),x(1,0,u),\dots,x(i,0,u))]\right]\right). \tag{3.4}$$

For $u \in S_\nu$ let $x(k,0,u)$ be the solution of (1.1) satisfying $x(0) = u$. For this solution, in view of (i), (vii)–(ix), we have

$$\|Tu\| \le \left\|G^{-1}\right\|(\|\ell\| + \|L\|QP) \le \nu,$$

i.e., $TS_\nu \subseteq S_\nu$. To show the continuity of T on S_ν, we let $\{u_m\} \subset S_\nu$, $u_0 \in S_\nu$ such that $\lim_{m\to\infty}\|u_m - u_0\| = 0$. We define $u^m = Tu_m$, and $u^0 = Tu_0$, then we have

$$\begin{aligned} \left\|u^m - u^0\right\| \le \left\|G^{-1}\right\|\|L\|Q\sum_{i=0}^{\infty}\|Y(i)\,[&f_i(x(0,0,u_m),x(1,0,u_m),\dots,x(i,0,u_m)) \\ &-f_i(x(0,0,u_0),x(1,0,u_0),\dots,x(i,0,u_0))]\,\|. \end{aligned} \tag{3.5}$$

From the continuity of f_i, it is clear that the summand in (3.5) tends to zero as $m \to \infty$, further in view of (viii) it is uniformly bounded by the summable function $2q(i)$. Thus, it follows from Lebesgue's dominated convergence theorem that $\|u^m - u^0\| \to 0$ as $m \to \infty$, i.e., T is continuous on S_ν. Thus, by Brouwer's fixed point theorem there exists at least one vector, x_0, such that equation (3.3) holds. The solution of (1.1) with this initial condition also satisfies the boundary condition (1.2). ∎

COROLLARY 3.4. *With respect to the boundary value problem (1.1),(1.2), assume that the operator L is defined on $B(\mathbb{N})$ and the conditions (i),(vi) hold. Further, let*

(x) $\liminf_{m\to\infty}(1/m)\sum_{i=0}^{\infty}\sup_{\|u\|\le m}\|Y(i)\,[b(i) + f_i(x(0,0,u),x(1,0,u),\dots,x(i,0,u))]\| = 0.$

Then, for every $\ell \in \mathbb{R}^n$ there exists at least one solution of (1.1),(1.2).

PROOF. The proof is similar to that of Corollary 3.2. ∎

In Theorem 3.1 functions x are taken from a suitable closed, convex, and bounded subset S^ν of the Banach space $B^\infty(\mathbb{N})$, and for the operator T defined in (3.1) it is shown that $TS^\nu \subset S^\nu$. However, the applicability of Schauder's fixed point theorem fails, at least as far as we can check, if there is no such S^ν. A similar remark holds for Theorem 3.3 also. One of the alternatives in such a case is the Leray-Schauder fixed point theorem. For this, we introduce a parameter $\mu \in [0,1]$ in the problem (1.1),(1.2) as follows:

$$\begin{aligned} x(k+1) &= \sum_{i=0}^{k} A_k(i)x(i) + \mu b(k) + \mu f_k(x(0),x(1),\dots,x(k)), \qquad k \in \mathbb{N}, \\ L[x] &= \mu\ell, \end{aligned} \tag{3.6}$$

so that the operator equation (3.1) becomes

$$\begin{aligned} [T(x,\mu)](k) = \mu\Bigg\{&X(k)\sum_{i=0}^{k-1} Y(i)\,[b(i) + f_i(x(0),x(1),\dots,x(i))] \\ &-\mathcal{A}(k)G^{-1}L\left[\sum_{i=0}^{k-1}\mathcal{B}_k(i)\,[b(i) + f_i(x(0),x(1),\dots,x(i))]\right] + \mathcal{A}(k)G^{-1}\ell\Bigg\}. \end{aligned} \tag{3.7}$$

We shall show that there exists a function $x \in B(\mathbb{N})$ such that $[T(x,1)](k) = x(k)$. For this, we shall use the following result, which is a particular case of [40, Theorem 1, p. 64].

THEOREM 3.5. *Let B be a Banach space. For the operator equation*

$$T(x,\mu) - x = 0 \tag{3.8_μ}$$

assume the following.

(I) *$T(x,\mu)$ is defined on $B \times [0,1]$ with values in B, and is completely continuous in x, i.e., for each $\mu \in [0,1]$, $T(x,\mu)$ is continuous in x and maps every bounded subset of B into a relatively compact set. Moreover, if Z is a bounded subset of B, $T(x,\mu)$ is continuous in μ uniformly with respect to $x \in Z$.*

(II) *$T(x,\mu_0) = 0$ for some $\mu_0 \in [0,1]$ and for every $x \in B$.*

(III) *If there are solutions of the equation $(3.8)_\mu$, then they belong to the same closed ball Σ of B, independently of μ.*

Then, there exists a continuum of solutions of $(3.8)_\mu$, corresponding to all the values of $\mu \in [0,1]$. All these solutions lie in Σ.

THEOREM 3.6. *With respect to the boundary value problem (1.1),(1.2), assume that the operator L is defined on $B(\mathbb{N})$ and Condition (i) holds. Further, let*

(xi) *for every $Z \subset B(\mathbb{N})$, $\sup_{x\in Z} \|Y(i)\,[b(i) + f_i(x(0),x(1),\dots,x(i))]\| \le s(i)\|x\|_Z + q(i)$, where for $x \in Z$, $\|x\|_Z = \sup_{k\in\mathbb{N}} \|x(k)\|$, and $\sum_{i=0}^{\infty} s(i) = W < \infty$, $\sum_{i=0}^{\infty} q(i) = P < \infty$;*

(xii) *$Q^2\|G^{-1}\|\|L\|We^{QW} < 1$.*

Then, for every $\ell \in \mathbb{R}^n$ there exists at least one solution of (1.1),(1.2).

PROOF. In view of Theorem 2.3 for $\mu = 0$ the problem (3.6) has only the trivial solution. Let $\{k_m\}$ be an increasing sequence of positive integers such that $k_m \to \infty$ as $m \to \infty$. Further, let $\mathbb{N}_m = \{0,1,\dots,k_m\}$, and $B(\mathbb{N}_m)$ be the space of all real n-vector functions defined on $\mathbb{N}_m$ with the norm $\|x\|_{B_m} = \sup_{k\in\mathbb{N}_m} \|x(k)\|$. Assume that $x \in B(\mathbb{N}_1)$ and consider the function

$$\overline{x}(k) = \begin{cases} x(k), & k \in \mathbb{N}_1 \\ x(k_1), & k \in \mathbb{N} - \mathbb{N}_1. \end{cases}$$

The set of all such functions $\overline{x}$ is a Banach space $D_1(\mathbb{N})$ with the norm $\|\overline{x}\|_{D_1} = \|x\|_{B_1}$. Now consider the operator $T_1(\overline{x},\mu) : D_1(\mathbb{N}) \to D_1(\mathbb{N})$ with $[T_1(\overline{x},\mu)](k) = \overline{y}(k)$, where for $k \in \mathbb{N}_1$,

$$\begin{aligned} y(k) = \mu \Bigg\{ & X(k) \sum_{i=0}^{k-1} Y(i)\,[b(i) + f_i(\overline{x}(0),\overline{x}(1),\dots,\overline{x}(i))] \\ & - \mathcal{A}(k)G^{-1}L\left[\sum_{i=0}^{k-1} \mathcal{B}_k(i)\,[b(i) + f_i(\overline{x}(0),\overline{x}(1),\dots,\overline{x}(i))]\right] + \mathcal{A}(k)G^{-1}\ell \Bigg\}. \end{aligned} \tag{3.9}$$

We shall establish a fixed point theorem for the operator $T_1(\overline{x},1)$. For this, first we shall prove the continuity of $T_1(\overline{x},\mu)$ with respect to μ. Let $\mu_1,\mu_2 \in [0,1]$ and $\overline{x} \in D_1(\mathbb{N})$. Then, for all $k \in \mathbb{N}_1$, we have

$$\begin{aligned} &\|[T_1(\overline{x},\mu_1)](k) - [T_1(\overline{x},\mu_2)](k)\| \\ &\qquad \le |\mu_1 - \mu_2| \left[Q\left(\|G^{-1}\|\left(\|\ell\| + Q\|L\|P\right) + P\right) + Q\left(Q\left\|G^{-1}\right\|\|L\| + 1\right)W\|x\|_{D_1}\right]. \end{aligned}$$

Consequently, since

$$\|T_1(\overline{x},\mu_1) - T_1(\overline{x},\mu_2)\|_{D_1} = \sup_{k\in\mathbb{N}_1} \|[T_1(\overline{x},\mu_1)](k) - [T_1(\overline{x},\mu_2)](k)\|,$$

it follows that the operator $T_1(\overline{x}, \mu)$ is continuous in μ uniformly with respect to $\overline{x} \in Z$ (any bounded subset of $D_1(\mathbb{N})$). To show the continuity of $T_1(\overline{x}, \mu)$ with respect to $\overline{x}$, we let $\{\overline{x}_m\}$, $\overline{x}$ in $D_1(\mathbb{N})$ and define $\overline{y}_m(k) = [T_1(\overline{x}_m, \mu)](k)$, $m = 1, 2, \ldots$, $\overline{y}(k) = [T_1(\overline{x}, \mu)](k)$ and assume that

$$\lim_{m\to\infty} \|\overline{x}_m - \overline{x}\|_{D_1} = \lim_{m\to\infty} \|x_m - x\|_{B_1} = 0.$$

Then, we have

$$\begin{aligned} \|\overline{y}_m - \overline{y}\|_{D_1} &= \sup_{k\in\mathbb{N}_1} \|y_m(k) - y(k)\| \\ &\le Q \sum_{i=0}^{\infty} \|Y(i) [f_i(\overline{x}_m(0), \ldots, \overline{x}_m(i)) - f_i(\overline{x}(0), \ldots, \overline{x}(i))]\| \\ &\quad + Q^2 \left\|G^{-1}\right\| \|L\| \sum_{i=0}^{\infty} \|Y(i) [f_i(\overline{x}_m(0), \ldots, \overline{x}_m(i)) - f_i(\overline{x}(0), \ldots, \overline{x}(i))]\| . \end{aligned} \tag{3.10}$$

However, since

$$\lim_{m\to\infty} \|f_i(\overline{x}_m(0), \ldots, \overline{x}_m(i)) - f_i(\overline{x}(0), \ldots, \overline{x}(i))\| = 0,$$

and for $i \in \mathbb{N}$

$$\|Y(i) [f_i(\overline{x}_m(0), \ldots, \overline{x}_m(i)) - f_i(\overline{x}(0), \ldots, \overline{x}(i))] \| \le s(i) [\|x_m\|_{\mathbb{N}_1} + \|x\|_{\mathbb{N}_1}] + 2q(i),$$

and also, the functions $s(i)$ and $q(i)$ are summable, from Lebesgue's dominated convergence theorem, (3.10) implies that $\lim_{m\to\infty} \|\overline{y}_m - \overline{y}\|_{D_1} = 0$. This proves the continuity of $T_1(\overline{x}, \mu)$ with respect to $\overline{x}$.

Also since

$$\|\overline{y}\|_{D_1} = \|y\|_{B_1} \le Q \left\|G^{-1}\right\| [\|\ell\| + Q\|L\| (Wb_Z + P)] + Q (Wb_Z + P),$$

$T_1(Z, \mu)$ is relatively compact in $D_1(\mathbb{N})$ for each $\mu \in [0, 1]$, (cf., see [18]).

Now assume that the equation

$$[T_1(\overline{x}, \mu)](k) - \overline{x}(k) = 0 \tag{3.11}$$

has solutions in $D_1(\mathbb{N})$. Then, if $\overline{x}(k)$ is such a solution corresponding to a fixed $\mu \in [0, 1]$, then for $k \in \mathbb{N}_1$, we have

$$\|x(k)\| \le Q \left\|G^{-1}\right\| [\|\ell\| + Q\|L\| (W\|\overline{x}\|_{D_1} + P)] + QP + Q \sum_{i=0}^{k-1} s(i)\|x(i)\|.$$

Thus, by Gronwall's inequality [17, p. 183] it follows that

$$\|x(k)\| \le \left[Q \left\|G^{-1}\right\| [\|\ell\| + Q\|L\| (W\|\overline{x}\|_{D_1} + P)] + QP\right] \prod_{i=0}^{k-1} (1 + Qs(i)),$$

which in view of $\prod_{i=0}^{k-1} (1 + Qs(i)) \le \exp(Q \sum_{i=0}^{\infty} s(i)) = e^{QW}$ gives

$$\|\overline{x}\|_{D_1} \le \left(1 - Q^2 \left\|G^{-1}\right\| \|L\| We^{QW}\right)^{-1} Q \left[\left\|G^{-1}\right\| (\|\ell\| + QP\|L\|) + P\right] e^{QW} = \omega.$$

Thus, the solutions of (3.11) are uniformly bounded with respect to $\mu \in [0, 1]$. Hence, the conditions of Theorem 3.5 are satisfied, and we conclude that the operator $T_1(\overline{x}, 1)$ has a fixed point $\overline{x}$, i.e., $[T_1(\overline{x}, \mu)](k) = \overline{x}(k)$ so $\overline{x} \in D_1(\mathbb{N})$.

Now let $D_m(\mathbb{N})$ be the Banach space of all functions $\overline{x}$, which are defined from the functions $x \in B(\mathbb{N}_m)$ as follows

$$\overline{x}(k) = \begin{cases} x(k), & k \in \mathbb{N}_m, \\ x(k_m), & k \in \mathbb{N} - \mathbb{N}_m, \end{cases}$$

with the norm $\|\overline{x}\|_{D_m} = \|x\|_{B_m} = \sup_{k\in N_m} \|x(k)\|$. Then, on following the above process, we can find a sequence $\{x_m\}$ of solutions of (1.1) such that $\overline{x}_m \in D_m(\mathbb{N})$, $\|\overline{x}_m\|_{D_m} = \|x_m\|_{B_m} \le \omega$, and for $k \in \mathbb{N}_m$,

$$\begin{aligned} x_m(k) = {} & X(k)\sum_{i=0}^{k-1} Y(i)\left[b(i) + f_i(\overline{x}_m(0), \dots, \overline{x}_m(i))\right] \\ & - \mathcal{A}(k)G^{-1}L\left[\sum_{i=0}^{k-1} \mathcal{B}_k(i)\left[b(i) + f_i(\overline{x}_m(0), \dots, \overline{x}_m(i))\right]\right] + \mathcal{A}(k)G^{-1}\ell. \end{aligned} \tag{3.12}$$

Now the sequence $\{\overline{x}_m\}$ is uniformly bounded on $\mathbb{N}_1$. The Arzela-Ascoli theorem [18] guarantees a subsequence S_1 of $\mathbb{N}$ and a $x^1 \in B(\mathbb{N}_1)$ with

$$\sup_{k\in\mathbb{N}_1} \left\|\overline{x}_m(k) - x^1(k)\right\| \to 0, \qquad \text{as } m \to \infty \text{ through } S_1.$$

Similarly there exists a subsequence S_2 of S_1 and a $x^2 \in B(\mathbb{N}_2)$ with

$$\sup_{k\in\mathbb{N}_2} \left\|\overline{x}_m(k) - x^2(k)\right\| \to 0, \qquad \text{as } m \to \infty \text{ through } S_2.$$

Notice also $x^1(k) = x^2(k)$ if $k \in \mathbb{N}_1$. Proceed inductively to obtain for $i = 1, 2 \dots$ a subsequence S_i of S_{i-1} and a $x^i \in B(\mathbb{N}_i)$ with

$$\sup_{k\in\mathbb{N}_i} \left\|\overline{x}_m(k) - x^i(k)\right\| \to 0, \qquad \text{as } m \to \infty \text{ through } S_i.$$

Also $x^i(k) = x^{i-1}(k)$ if $k \in \mathbb{N}_{i-1}$. Define $x : \mathbb{N} \to \mathbb{R}^n$ as follows. Fix $c \in \mathbb{N}$ and let $j \in \mathbb{N}$ with $c \le k_j$. Let $x(c) = x^j(c)$. Now x is well defined and $\|x\|_B \le \omega$. Fix c and choose $k_j \ge c$. Then for $m \in S_j - N_j$ and $k \in \mathbb{N}_j$, we have (from (3.1)) that

$$\begin{aligned} \overline{x}_m(k) = {} & X(k)\sum_{i=0}^{k-1} Y(i)\left[b(i) + f_i(\overline{x}_m(0), \dots, \overline{x}_m(i))\right] \\ & - \mathcal{A}(k)G^{-1}L\left[\sum_{i=0}^{k-1} \mathcal{B}_k(i)\left[b(i) + f_i(\overline{x}_m(0), \dots, \overline{x}_m(i))\right]\right] + \mathcal{A}(k)G^{-1}\ell. \end{aligned}$$

Let $m \to \infty$ through S_j (using the Lebesgue dominated convergence theorem) to obtain

$$\begin{aligned} x^j(k) = {} & X(k)\sum_{i=0}^{k-1} Y(i)\left[b(i) + f_i(x^j(0), \dots, x^j(i))\right] \\ & - \mathcal{A}(k)G^{-1}L\left[\sum_{i=0}^{k-1} \mathcal{B}_k(i)\left[b(i) + f_i(x^j(0), \dots, x^j(i))\right]\right] + \mathcal{A}(k)G^{-1}\ell. \end{aligned}$$

Thus,

$$\begin{aligned} x(k) = {} & X(k)\sum_{i=0}^{k-1} Y(i)\left[b(i) + f_i(x(0), \dots, x(i))\right] \\ & - \mathcal{A}(k)G^{-1}L\left[\sum_{i=0}^{k-1} \mathcal{B}_k(i)\left[b(i) + f_i(x(0), \dots, x(i))\right]\right] + \mathcal{A}(k)G^{-1}\ell \end{aligned}$$

for $k \in \mathbb{N}_j$ (in particular for $k = c$). Since c is arbitrary the result follows. ∎

COROLLARY 3.7. *With respect to the boundary value problem (1.1), (1.2) assume that the operator L is defined on $B^\infty(\mathbb{N})$ and in addition to Condition C_1 the hypotheses of Theorem 3.6 hold. Then, for every $\ell \in \mathbb{R}^n$ there exists at least one solution of (1.1),(1.2) in $B^\infty(\mathbb{N})$.*

4. NONLINEAR PROBLEMS WITH A PARAMETER

Here, we shall prove the following constructive result for the boundary value problem (1.3),(1.2).

THEOREM 4.1. *With respect to the boundary value problem (1.3),(1.2) assume that the operator L is defined on $B(\mathbb{N})$, $\xi \in S_\nu$ and Condition (i) holds. Further, let*

(a) *for all x, $\bar{x} \in S^\nu$, and λ, $\bar{\lambda} \in S_\nu$, $\left\|Y(i)\left[g_i(x(0),\dots,x(i),\lambda) - g_i(\bar{x}(0),\dots,\bar{x}(i),\bar{\lambda})\right]\right\| \le \theta(i)(\|x-\bar{x}\|_B + \|\lambda - \bar{\lambda}\|)$, where $\sum_{i=0}^\infty \theta(i) = C < \infty$;*

(b) $$\sum_{i=0}^{\infty} \sup_{\substack{x \in S^\nu \\ \lambda \in S_\nu}} \|Y(i)\left[b(i) + g_i(x(0), x(1), \dots, x(i), \lambda)\right]\| = P < \infty, \quad \text{and} \quad Q(\|\xi\|+P) \le \nu;$$

(c) *for every $x \in S^\nu$ such that $x(0) = \xi$ and $\lambda, \bar{\lambda} \in S_\nu$,*

$$\left\| L\left[X(k)\sum_{i=0}^{k-1} Y(i)\left[g_i(x(0),x(1),\dots,x(i),\lambda) - g_i(x(0),x(1),\dots,x(i),\bar{\lambda})\right]\right]\right\| \ge \Lambda\|\lambda - \bar{\lambda}\|,$$

and

$$QC\left(1 + \frac{\|L\|QC}{\Lambda}\right) < 1; \tag{4.1}$$

(d) *for a fixed $\lambda \in S_\nu$ there exists a function $x \in S^\nu$, and for each $x \in S^\nu$ there exists a $\lambda \in S_\nu$ such that the solution $u(k)$ of the system*

$$u(k+1) = \sum_{i=0}^{k} A_k(i)u(i) + b(k) + g_k(x(0), x(1), \dots, x(k), \lambda), \qquad k \in \mathbb{N},$$

with $u(0) = \xi$ satisfies $L[u] = \ell$.

Then, the problem (1.3),(1.2) has at least one solution.

PROOF. Let λ_0 be a vector in S_ν and x_0 be a function in S^ν such that the function

$$x_1(k) = \mathcal{A}(k)\xi + X(k)\sum_{i=0}^{k-1} Y(i)\left[b(i) + g_i(x_0(0), x_0(1), \dots, x_0(i), \lambda_0)\right]$$

satisfies $L[x_1] = \ell$. The existence of such a function $x_1(k)$ is guaranteed by the condition (d), and for this function in view of (b), we have

$$\|x_1\|_B \le Q\|\xi\| + QP = Q(\|\xi\| + P) \le \nu,$$

i.e., $x_1 \in S^\nu$. Now, from Conditions (d) and (b), it is clear that the iterative scheme

$$x_m(k) = \mathcal{A}(k)\xi + X(k)\sum_{i=0}^{k-1} Y(i)\left[b(i) + g_i(x_{m-1}(0), x_{m-1}(1), \dots, x_{m-1}(i), \lambda_{m-1})\right]$$

is well defined, i.e., for all $m \ge 1$, $\lambda_{m-1} \in S_\nu$, $x_{m-1} \in S^\nu$ and $L[x_m] = \ell$. Further, we have

$$\begin{aligned} &\|x_m(k) - x_{m-1}(k)\| \\ &\le \|X(k)\| \sum_{i=0}^{k-1} \|Y(i)\left[g_i(x_{m-1}(0), \dots, x_{m-1}(i), \lambda_{m-1}) - g_i(x_{m-2}(0), \dots, x_{m-2}(i), \lambda_{m-2})\right]\| \\ &\le QC\left(\|x_{m-1} - x_{m-2}\|_B + \|\lambda_{m-1} - \lambda_{m-2}\|\right). \end{aligned} \tag{4.2}$$

Moreover, since for each $m \geq 1$,

$$\ell = L[x_m] = L[\mathcal{A}(k)\xi] + L\left[X(k)\sum_{i=0}^{k-1} Y(i)\left[b(i) + g_i(x_{m-1}(0), x_{m-1}(1), \dots, x_{m-1}(i), \lambda_{m-1})\right]\right]$$

in view of (c) and (a), it follows that

$$\begin{aligned} 0 = &\left\| L\left[X(k)\sum_{i=0}^{k-1} Y(i)\left[g_i(x_{m-1}(0), \dots, x_{m-1}(i), \lambda_{m-1}) - g_i(x_{m-1}(0), \dots, x_{m-1}(i), \lambda_{m-2})\right]\right] \right. \\ &\left. + L\left[X(k)\sum_{i=0}^{k-1} Y(i)\left[g_i(x_{m-1}(0), \dots, x_{m-1}(i), \lambda_{m-2}) - g_i(x_{m-2}(0), \dots, x_{m-2}(i), \lambda_{m-2})\right]\right] \right\| \\ \geq\, &\Lambda \|\lambda_{m-1} - \lambda_{m-2}\| - \|L\| QC \|x_{m-1} - x_{m-2}\|_B, \end{aligned}$$

or,

$$\|\lambda_{m-1} - \lambda_{m-2}\| \leq \frac{\|L\| QC}{\Lambda} \|x_{m-1} - x_{m-2}\|_B. \tag{4.3}$$

On combining (4.2) and (4.3), we obtain

$$\|x_m - x_{m-1}\|_B \leq QC\left(1 + \frac{\|L\|QC}{\Lambda}\right) \|x_{m-1} - x_{m-2}\|_B,$$

which by induction leads to

$$\|x_m - x_{m-1}\|_B \leq (QC)^{m-1}\left(1 + \frac{\|L\|QC}{\Lambda}\right)^{m-1} \|x_1 - x_0\|_B.$$

Thus, in view of (4.1), it follows that $\lim_{m\to\infty} \|x_m - x\|_B = 0$, and from (4.3), $\lim_{m\to\infty} \|\lambda_m - \lambda\| = 0$, where $x \in S^\nu$, $x(0) = \xi$, and $\lambda \in S_\nu$.

Finally, let

$$u(k) = \mathcal{A}(k)\xi + X(k)\sum_{i=0}^{k-1} Y(i)\left[b(i) + g_i(x(0), x(1), \dots, x(i), \lambda)\right].$$

Then, an application of Lebesgue's dominated convergence theorem leads to $\lim_{m\to\infty} \|x_m - u\|_B = 0$, which shows that $u(k) \equiv x(k)$, $k \in \mathbb{N}$. This completes the proof of our theorem. ∎

REFERENCES

1. R.P. Agarwal and D. O'Regan, Multipoint boundary value problems for general discrete systems: The degenerate case, *Communications in Appl. Anal.* (to appear).
2. R.P. Agarwal and M. Sambandham, Multipoint boundary value problems for general discrete systems, *Neural, Parallel & Scientific Computations* (to appear).
3. P.Y.H. Pang and R.P. Agarwal, Monotone iterative methods for a general class of discrete boundary value problems, In *Advances in Difference Equations*, Special Issue of *Computers Math. Applic.* **28** (1–3), 243–254 (1994).
4. P.Y.H. Pang and R.P. Agarwal, Asymptotic behaviour of a general class of difference systems, *Mathl. Comput. Modelling* **22** (3), 39–47 (1995).
5. P.Y.H. Pang and R.P. Agarwal, On periodicity of difference equations of a general type, *J. Difference Equations and Appl.* **2**, 271–287 (1996).
6. P.Y.H. Pang and R.P. Agarwal, Stability of a general class of difference systems, In *Advances in Difference Equations II*, Special Issue of *Computers Math. Applic.* (to appear).
7. P.Y.H. Pang and R.P. Agarwal, Symmetries of difference systems on manifolds, *Dynamic Systems and Applications* (to appear).
8. R.P. Agarwal and L.E. Shaikhet, Linear square optimal control problem for stochastic difference equations with unknown parameters, (to appear).

9. M.R. Crisci, V.B. Kolmanovskii, E. Russo and A. Vecchio, Stability of difference Volterra equations: Direct Liapunov method and numerical procedure, In *Advances in Difference Equations II*, Special Issue of *Computers Math. Applic.* (to appear).
10. V.B. Kolmanovskii and L.E. Shaikhet, General method of Lyapunov functionals construction for stability investigations of stochastic difference equations, In *Dynamical Systems and Applications*, WSSIAA 4, pp. 397–439, (1995).
11. N.V. Kuchkina and L.E. Shaikhet, Linear-square optimal control problem for stochastic difference second kind Volterra equations, In *Advances in Difference Equations II*, Special Issue of *Computers Math. Applic.* (to appear).
12. R.P. Agarwal, On multipoint boundary value problems for discrete equations, *J. Math. Anal. Appl.* **96**, 520–534 (1983).
13. R.P. Agarwal, Initial-value methods for discrete boundary value problems, *J. Math. Anal. Appl.* **100**, 513–529 (1984).
14. R.P. Agarwal, Computational methods for discrete boundary value problems, *Appl. Math. Comp.* **18**, 15–41 (1986).
15. R.P. Agarwal, Computational methods for discrete boundary value problems II, *J. Math. Anal. Appl.* **166**, 540–562 (1992).
16. R.P. Agarwal, Component-wise convergence of quasilinear method for nonlinear boundary value problems, *Hiroshima Math. J.* **22**, 525–541 (1992).
17. R.P. Agarwal, *Difference Equations and Inequalities*, Marcel Dekker, New York, (1992).
18. R.P. Agarwal and D. O'Regan, A fixed point approach for nonlinear discrete boundary value problems, In *Advances in Difference Equations II*, Special Issue of *Computers Math. Applic.* (to appear).
19. R.P. Agarwal and D. O'Regan, Boundary value problems for discrete equations, *Appl. Math. Lett.* (to appear).
20. P.W. Eloe, Difference equations and multipoint boundary value problems, *Proc. Amer. Math. Soc.* **86**, 253–259 (1982).
21. P.W. Eloe, A boundary value problem for a system of difference equations, *Nonlinear Analysis* **7**, 813–820 (1983).
22. D.L. Etheridge and J. Rodriguez, On perturbed discrete boundary value problems, (preprint).
23. J. Henderson and B.A. Lawrence, Smooth dependence on boundary matrices, *J. Difference Equations* **2**, 161–166 (1996).
24. M. Kwapisz, Some existence and uniqueness results for boundary value problems for difference equations, *Applicable Analysis* **37**, 169–182 (1990).
25. M. Kwapisz, On boundary value problems for difference equations, *J. Math. Anal. Appl.* **156**, 254–270 (1991).
26. M. Kwapisz, On modification of Samoilenko's numerical-analytic method for solving boundary value problems for difference equations, *Applications of Mathematics* **38**, 133–144 (1993).
27. K.N. Murty and P.V.S. Anand, Periodic multipoint boundary value problems associated with a first order nonlinear Liapunov type matrix difference system—Existence and uniqueness, *Int. J. Non-linear Diff. Eqns.* **1**, 197–214 (1995).
28. J. Rodriguez, On nonlinear discrete boundary value problems, *J. Math. Anal. Appl.* **114**, 398–408 (1986).
29. J. Rodriguez, On resonant discrete boundary value problem, *Applicable Analysis* **19**, 265–274 (1985).
30. J. Rodriguez, Resonance in nonlinear discrete systems with constraints, In *Proceedings of 24^{th} Conference on Decision and Control*, December 1985, Ft. Lauderdale, FL, IEEE, pp. 1738–1743.
31. R. Shridharan and R.P. Agarwal, Stationary and nonstationary iterative methods for nonlinear boundary value problems, *Mathl. Comput. Modelling* **18** (2), 43–62 (1993).
32. R. Shridharan and R.P. Agarwal, General iterative methods for nonlinear boundary value problems, *J. Austral. Math. Soc. Ser. B* **37**, 58–85 (1995).
33. M. Urabe, An existence theorem for multi-point boundary value problems, *Funkcialaj Ekvacioj* **9**, 43–60 (1966).
34. M. Urabe, The Newton method and its application to boundary value problems with nonlinear boundary conditions, In *Proc. U.S.-Japan Seminar on Differential and Functional Equations*, Benjamin, pp. 383–410, (1967).
35. M. Urabe, The degenerate case of boundary value problems associated with weakly nonlinear differential systems, *Pub. Resh. Inst. Mathl. Sci.*, Kyoto Univ. **4**, 545–584 (1968).
36. T. Yamamoto, An existence theorem of solution to boundary value problems and its applications to error estimates, *Math. Japonica* **27**, 301–318 (1982).
37. A.G. Kartsatos, A boundary value problem on an infinite interval, *Proc. Edin. Math. Soc.* **19**, 245–252 (1974/75).
38. A.G. Kartsatos, The Leray-Schauder theorem and the existence of solutions to boundary value problems on infinite intervals, *Indiana Univ. Math. J.* **23**, 1021–1029 (1974).
39. C. Avramescu, Sur l' éxistence des solutions convergentes d' équations différentielles non linéaires, *Ann. Mat. Pura Appl.* **4**, 147–168 (1969).
40. J. Leray and J. Schauder, Topologie et équations fonctionnelles, *Ann. Ec. Norm. Sup., S. III* **51**, 45–78 (1934).